Construction of a Penrose Diagram for an Accreting Black Hole

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Abstract

A Penrose diagram is constructed for a spatially coherent black hole that accretes at stepwise steady rates as measured by a distant observer from an initial state described by a metric of Minkowski form. Coordinate lines are computationally derived, and radial light-like trajectories verify the viability of the diagram. Coordinate dependencies of significant features, such as the horizon and radial mass scale, are clearly demonstrated on the diagram. The onset of a singularity at the origin is shown to open a new region in space-time that contains the interior of the black hole.

1   Introduction

Astrophysical black holes are expected to evolve, i.e. accrete or evaporate, over time. One expects that the initiation and final evaporation processes of black hole dynamics should involve spatial coherence on scales comparable to those describing the medium scale structure of the space-time, since the geometries during those periods are defined by micro-physics. It is of particular interest to examine the correspondence of low-curvature space-time with a dynamic black hole geometry.

The non-orthogonal temporal coordinate associated with the river model of black holes[1, 2] has been shown to provide a convenient parameter for describing the evolution of a black hole without physical singularities in the vicinity of the horizon. In a previous paper, we examined an example dynamic black hole undergoing a steady rate of evaporation[3]. To complement evaporation, in this paper we will examine the global causal structure of a black hole undergoing periods of stepwise steady accretion. This is accomplished via the construction of a computationally derived Penrose diagram. The viability of this diagram can be verified by computing and plotting radial light-like trajectories as curves of slope ±unity. The complete evolutionary cycle of a black hole should involve a merger of the techniques involved in these companion papers[4].

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2 Form of the Metric and Conformal Coordinates

2.1 Form of the metric

The dynamic space-time metric will be assumed to take the form

$$ds^2 = -\left(1 - \frac{R_M(ct)}{r}\right)(dct)^2 + 2\sqrt{\frac{R_M(ct)}{r}}dct\ dr + dr^2 + r^2\left(d\theta^2 + sin^2\theta\ d\phi^2\right). \quad (2.1)$$

In this equation, $R_M(ct) \equiv 2G_N M(ct)/c^2$ is a time-dependent form of the Schwarzschild radius that we refer to as the *radial mass scale*. This metric was examined as a dynamic extension of the river model of (static) black holes discussed in the literature[1, 5]. The metric takes the form of a Minkowski space-time both asymptotically $(r \to \infty)$ as well as when the radial mass scale vanishes $(R_M(ct) \to 0)$. Therefore, the temporal and radial coordinates are those of an observer far from the black hole. All curvature components generated by this metric are non-singular near the light-like surface defining the horizon, $\dot{R}_H = 1 - \sqrt{\frac{R_M}{R_H}}$, as well as the surface defined by the radial mass scale, $R_M(ct)$. This simplifies descriptions of the physics in the vicinity of R_H and R_M.

2.2 Form of conformal coordinates

The form of the conformal temporal and radial coordinates that are used for the construction of the Penrose diagram have been developed in companion papers[5, 3]:

$$ct_* = \frac{r}{2}\left(exp\left[\int^{\frac{R_M(ct)}{r}} \frac{\left(1+\sqrt{\zeta'}\right)d\zeta'}{\left\{\zeta'\left(1+\sqrt{\zeta'}\right)+\dot{R}_M\right\}}\right] - exp\left[\int^{\frac{R_M(ct)}{r}} \frac{\left(1-\sqrt{\zeta'}\right)d\zeta'}{\left\{\zeta'\left(1-\sqrt{\zeta'}\right)-\dot{R}_M\right\}}\right]\right)$$

$$r_* = \frac{r}{2}\left(exp\left[\int^{\frac{R_M(ct)}{r}} \frac{\left(1+\sqrt{\zeta'}\right)d\zeta'}{\left\{\zeta'\left(1+\sqrt{\zeta'}\right)+\dot{R}_M\right\}}\right] + exp\left[\int^{\frac{R_M(ct)}{r}} \frac{\left(1-\sqrt{\zeta'}\right)d\zeta'}{\left\{\zeta'\left(1-\sqrt{\zeta'}\right)-\dot{R}_M\right\}}\right]\right).$$

$$(2.2)$$

The transformations are valid for constant accretion rates $\ddot{R}_M = 0$. These equations relate the space-time coordinates of an asymptotic observer (ct, r) with the conformal coordinates (ct_*, r_*). In subsequent calculations, the constants associated with the integrations are chosen so that the conformal space-time parameters (ct_*, r_*) asymptotically (i.e., as $\frac{R_M(ct)}{r} \to 0$) correspond to Minkowski space-time parameters (ct, r), which are likewise conformal.

3   Penrose Diagram of the Accreting Black Hole

3.1   Procedure for constructing the Penrose diagram

The diagram that is developed uses hyperbolic tangents of a scaled multiple of the conformal coordinates in Eq. 2.2 to map the infinite domain of those conformal coordinates onto a finite region. More specifically, the vertical coordinate Y_{t*} takes the form $(tanh(\frac{ct_*+r_*}{scale}) + tanh(\frac{ct_*-r_*}{scale}))/2$ and the horizontal coordinate Y_{r*} takes the form $(tanh(\frac{ct_*+r_*}{scale}) - tanh(\frac{ct_*-r_*}{scale}))/2$ in the exterior region. In the interior region $r < R_H(ct)$, the time-like and space-like coordinates (Y_{t*} and Y_{r*}) are interchanged (because of a sign change in the coefficient of $[-(dct_*)^2 + (dr_*)^2]$ in the conformal metric as one crosses the horizon) and shifted in a manner that preserves the coordinates of the horizon. The overall transformation assures that the slopes of *any* outgoing/ingoing light-like radial trajectories on the diagram will be ± 1, and that the diagram has its domain and range bounded by ± 1.

The initial metric will be assumed to be of the form of Minkowski space-time for $t \leq 0$. The accretion begins at $t = 0$ from an initial radial mass scale $R_M(0) = 0$, which smoothly transitions the dynamic metric in Eq. 2.1 from a Minkowski metric form. The radial coordinates must smoothly match across the volume $t = 0$, since $4\pi r^2$ measures the area of any sphere of radial coordinate r in both metric forms. Since the conformal coordinates developed require a steady rate of accretion, changes in that rate (if desired) can be modeled using stepwise steady accretions, and matching the coordinates of the radial mass scale R_M (a physical parameter directly related to the mass of the black hole) across differing rates with instantaneously matching metrics.

3.2   Features of the Penrose diagram

The Penrose diagram in Figure 1 demonstrates the expected global structure of this spherically symmetric black hole that accretes at a steady rate of change in the radial mass scale $R_M(ct)$ with respect to the distant observer's time coordinate ct. In the diagram, the red curves that are time-like in the right hand regions represent curves of constant r, originally graded from $r = 0$ in hundredths, tenths, then in multiples, and decades of the chosen scale. The curves of constant radial coordinate r all originate at the bottom corner of the diagram representing $t = -\infty$, and terminate at the uppermost corner representing $t = +\infty$. The green curves that are space-like in the right hand regions represent curves of constant ct graded in multiples of the given scale. All constant ct curves originate on the curve $r = 0$ and terminate at the far right corner of the diagram representing $r = \infty$. The various $ct = constant$ and $r = constant$ curves each intersect at only one point on the diagram. The light-like bounding curves $r = \infty$ on the right are
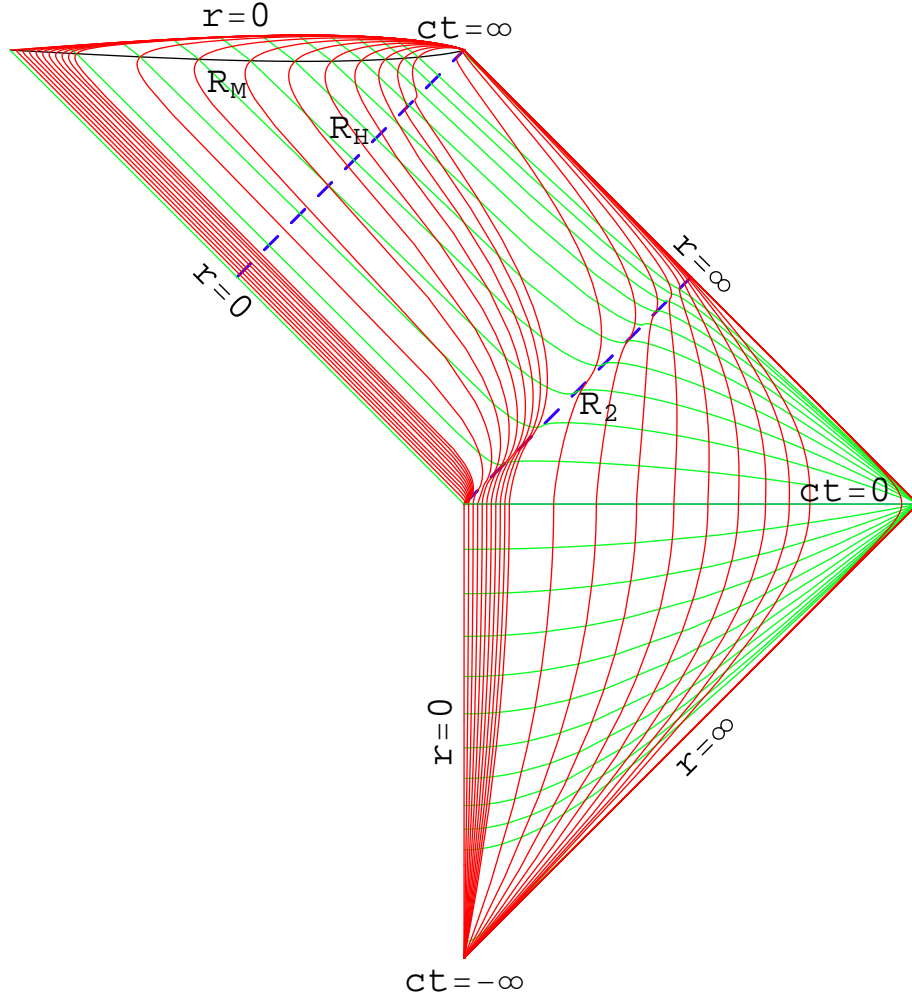
Figure 1: Penrose diagram for a black hole that accretes steadily from zero mass at $ct = 0$. Red curves (running vertically in the exterior right hand region) represent curves of constant r. The green curves represent curves of constant ct. The dashed blue line labeled R_H represents the horizon.

those of a Minkowski space-time.

Prior to the formation of the black hole, the space-time is taken to be that described by Minkowski, represented by the lower portion of the diagram. At $t = 0$, the singularity develops via an ingoing light-like transition ($ct = 0, r = 0$), during which a new region in the space-time opens up. This new region, which contains the interior of the black hole, is represented by the upper left quadrant in Figure 1. The curve $r = 0$ is initially a time-like trajectory bounding the Minkowski space-time on the left. It then undergoes the light-like transition to become a space-like trajectory bounding the upper interior region of the black hole from above. The Penrose coordinates of the singularity $r = 0$ somewhat matches the behavior in the vertical Penrose coordinate of a static Schwarzschild singularity $Y^{Schwarzschild}_{singularity *} = +1$. For an accreting black hole, the radial mass scale $R_M(ct)$ (indicated by the solid black space-like curve between the singularity and the horizon R_H) can be shown generally to lie within the horizon[5], and it is seen to mirror the behavior of the singularity $r = 0$. Radial coordinate curves are seen to transition from time-like to space-like as they cross the radial mass scale. This occurs because outgoing light-like trajectories, which on this geometry satisfy $\dot{r}_\gamma = 1 - \sqrt{R_M/r_\gamma}$, are temporarily stationary in the radial coordinate $\dot{r}_\gamma = 0$ at the radial mass scale.

The mass scale and horizon of the black hole are both dynamic. The horizon is represented by the dashed diagonal blue line indicated by R_H in Figure 1, and it lies completely within the new region of space-time opened up by the formation of the black hole singularity. Unlike as is the case for a static Schwarzschild black hole, the horizon $R_H(ct)$ is *not* a $t = \infty$ surface. Both radial and temporal coordinate curves are seen to cross the horizon and radial mass scale on the diagram. Thus, the distant observer's time is qualitatively different from that of a distant Schwarzschild observer, even for a very slowly accreting black hole.

Finally, there is a region of significant coordinate distortion along the light-like curve labeled R_2, which corresponds to one of the singular curves of the conformal metric[4]. The coefficient of the factor $[-(dct_*)^2 + (dr_*)^2]$ in the metric becomes singular on the outgoing light-like curve $R_2(ct)$ and vanishes on the horizon $R_H(ct)$. However, this anomalous coordinate behavior near R_2 does not correspond to any physical singularity on the geometry (for example, the Ricci scalar of the geometry, which is of the form $\mathcal{R} = \frac{3\dot{R}_M}{2r^2}\sqrt{\frac{r}{R_M}}$, is non-singular near R_2). For the steadily accreting black hole, in the distant future the area of the horizon of the black hole is unbounded ($R_H \to \infty$ as $t \to \infty$). However, as was the case for the steadily excreting black hole[3], it remains possible to stay external to the black hole, but not with a fixed radial coordinate r. A radial coordinate that is a fixed multiple of that of the horizon remains external if that multiple is greater than unity.

4 Conclusions and Discussion

The large-scale geometry of a steadily accreting black hole revealed several aspects of interest. The radial mass scale was seen to have a trajectory internal to the horizon of the black hole as expected from analytical examinations. The diagrammatic representation of the speed of light formation of the singularity was unexpected, but necessary, since low curvature space-times have non-negative conformal radial coordinates. This light-like formation of the singularity concurrently expands the large-scale structure of the space-time. Both the radial mass scale and the horizon were seen to lie completely within the region of space-time opened during the formation of the singularity.

The observation that the radial mass scale does not coincide with the horizon demonstrates that the dynamics described using our temporal coordinate t is qualitatively different from what would result from a mass scale dependency upon a Schwarzschild observer's temporal coordinate, no matter how small the rate of accretion. Parametric descriptions of both the radial mass scale and the horizon are non-singular in the coordinates (ct, r). In addition, all physical curvatures are well defined near these trajectories.

The asymptotic correspondence of the black hole description was seen to fix the scales of the past and future light-like infinities of the black hole to be the same as those of Minkowski space-time. Using the coordinates employed for the spatially coherent accreting black hole, when the accretion initiates there is continuity in the metric with that of static Minkowski space, but a discontinuity in curvature. The primary concerns of the present paper has been the construction of the Penrose diagram rather than the development of a model for the micro-physics describing the initial stage of accretion, which likely occurs on time scales small compared to the resolution of the diagram. Since the metrics precisely match at the transition, it is felt that the large scale structure has been accurately portrayed.

A Penrose diagram representing the complete life cycle of a model black hole can likewise be constructed using stepwise steady rates of accretion/evaporation, and matching metric forms across the transitions. A diagram demonstrating the initiation of accretion from a low curvature space-time, an end of accretion and beginning of evaporation, and an end of evaporation as the black hole mass vanishes, is presently being examined by the authors, and its form will be presented in a future manuscript. We ultimately want to examine the micro-physics that allows the initial accretion and final evaporation processes of a black hole. This will involve exploring general dynamics in the mass scale $R_M(ct)$ in a geometry with rates which can smoothly transition from and to low curvature space-times.

A. Brown, the co-author, unexpectedly passed during the preparation of the final draft of this manuscript.

References

[1] "The river model of black holes", A.J.S. Hamilton and J.P. Lisle, gr-qc/0411060 (2004) 14 pages.

[2] "Production and decay of evolving horizons", A.B. Nielsen and M. Visser, gr-qc/0510083 (2006) 25 pages, *Classical and Quantum Gravity* **23** (2006) 4637-4658.

[3] "Construction of a Penrose Diagram for a Spatially Coherent Evaporating Black Hole", B. A. Brown and J. Lindesay, arXiv:0710.2032v1 [gr-qc] (2007) 12 pages. *Class. Quantum Grav.* **25** (2008) 105026 (8pp) doi:10.1088/0264-9381/25/10/105026.

[4] "An Exploration of the Physics of Spherically Symmetric Dynamic Horizons", J. Lindesay Invited talk presented at the Joint Annual Conference of the National Society of Black Physicists and National Society of Hispanic Physicists (20-24 February 2008), Washington, DC, 22 February 2008, arXiv:0803.3018 [gr-qc] 25 pages.

[5] "Coordinates with non-singular curvature for a time-dependent black hole horizon", J. Lindesay, gr-qc/0609019 (2006), *Foundations of Physics* online DOI 10.1007/s10701-007-9146-4, 15 May 2007, 16 pages,.